1. Fundamental Relation and Polar Forms

The derivation begins with the fundamental relationship between the functions f_q, f_k, and g:

$$f_q(x_q, m)\,\overline{f_k(x_k, n)} = g(x_q, x_k,\, n - m) \tag{1}$$

The complex functions are expressed in their polar forms:

$$f_q(x_q, m) = R_q(x_q, m)\, e^{\,i\, O_q(x_q, m)} \tag{2}$$
$$f_k(x_k, n) = R_k(x_k, n)\, e^{\,i\, O_k(x_k, n)} \tag{3}$$
$$g(x_q, x_k, t) = R_g(x_q, x_k, t)\, e^{\,i\, O_g(x_q, x_k, t)} \tag{4}$$

2. Magnitude and Phase Equations

Substituting the polar forms into the main equation and separating the magnitude and phase gives two relations.

$$R_q(x_q, m)\, R_k(x_k, n) = R_g(x_q, x_k,\, n - m) \tag{5}$$
$$O_k(x_k, n) - O_q(x_q, m) = O_g(x_q, x_k,\, n - m) \tag{6}$$

Note: The phase relation (6) is taken as the starting point, as per your notes. It results from defining the phase of the product $f_q\overline{f_k}$ as $O_k - O_q$.

3. Analysis of the Base Case ($m = n$)

For the special case where $m = n$, the shift $t = n - m = 0$. The base functions (at zero shift) are:

$$f_q(x_q, 0) = \|q\|\, e^{\,i\, O_q} \tag{7}$$
$$f_k(x_k, 0) = \|k\|\, e^{\,i\, O_k} \tag{8}$$

where $\|q\|$ and $\|k\|$ are the base magnitudes, and O_q and O_k are the base phases.

Setting $m = n$ in equations (5) and (6), and using the zero-shift values $g(x_q, x_k, 0) = f_q(x_q, 0)\overline{f_k(x_k, 0)}$, we find the relationships for any shift m:

$$R_q(x_q, m)\, R_k(x_k, m) = R_g(x_q, x_k, 0) = \|q\|\|k\| \tag{9}$$
$$O_k(x_k, m) - O_q(x_q, m) = O_g(x_q, x_k, 0) = O_k - O_q \tag{10}$$

4. Deriving the Magnitude and Phase Functions

Magnitude

From equation (9), the product of the magnitudes is constant. The simplest solution consistent with the base case definitions is that the magnitudes are independent of the shift parameter m:

$$R_q(x_q, m) = R_q(x_q, 0) = \|q\| \tag{11}$$
$$R_k(x_k, m) = R_k(x_k, 0) = \|k\| \tag{12}$$

Phase

By rearranging the phase relation from (10), we get:

$$O_q(x_q, m) - O_q = O_k(x_k, m) - O_k$$

The left side depends only on (x_q, m) and the right on (x_k, m). For this to hold for all x_q, x_k, both sides must equal a function of only the common variable, m. We define this common phase shift as $\phi(m)$:

$$\phi(m) \equiv O_q(x_q, m) - O_q = O_k(x_k, m) - O_k \tag{13}$$

From this definition, we see that the base case is satisfied: $\phi(0) = O_q(x_q, 0) - O_q = 0$.

As shown in your notes, by setting $n = m + 1$ in the general phase relation (6), we find that the difference $\phi(m + 1) - \phi(m)$ is constant. This proves that $\phi(m)$ is an arithmetic progression. The general form is $\phi(m) = m\Theta + \gamma$. Since we know $\phi(0) = 0$, the offset γ must be zero. Therefore:

$$\phi(m) = m\Theta \tag{14}$$

This gives the final forms for the phase functions:

$$O_q(x_q, m) = O_q + m\Theta \tag{15}$$
$$O_k(x_k, n) = O_k + n\Theta \tag{16}$$

5. Final Solutions

From the preceding steps of the analysis, we established the forms for the magnitude and phase of the functions.

For $f_q(x_q, m)$:

- The magnitude is constant: $R_q(x_q, m) = \|q\|$.
- The phase is an arithmetic progression: $O_q(x_q, m) = O_q + m\Theta$.

Combining these gives the full expression for $f_q(x_q, m)$:

$$f_q(x_q, m) = R_q(x_q, m)\, e^{iO_q(x_q,m)} \tag{17}$$
$$= \|q\|\, e^{i(O_q+m\Theta)} \tag{18}$$

We can separate the exponential term using the property $e^{a+b} = e^a e^b$:

$$f_q(x_q, m) = \|q\|\, e^{iO_q}\, e^{im\Theta} \tag{19}$$

Now, we recognize that the term $\|q\|e^{iO_q}$ is simply the definition of the base function at zero shift, which is represented by the complex vector q:

$$q \equiv f_q(x_q, 0) = \|q\|e^{iO_q} \tag{20}$$

Substituting this definition back gives the final, compact form for f_q:

$$\mathbf{f_q(x_q, m) = q\, e^{im\Theta}} \tag{21}$$

The exact same logic is applied to derive the form for $f_k(x_k, n)$:

$$f_k(x_k, n) = R_k(x_k, n)\, e^{iO_k(x_k, n)} \tag{22}$$

$$= \|k\|\, e^{i(O_k + n\Theta)} \tag{23}$$

$$= \left(\|k\|\, e^{iO_k}\right) e^{in\Theta} \tag{24}$$

Defining the base vector k as $k \equiv f_k(x_k, 0) = \|k\|e^{iO_k}$, we arrive at the final solution for f_k:

$$\mathbf{f_k(x_k, n) = k\, e^{in\Theta}} \tag{25}$$